

# The Analytical Solution of Radiation Transfer Equation for a Layer of Magnetized Plasma With Random Irregularities

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## Abstract

The problem of radio wave reflection from an optically thick plane monotonous layer of magnetized plasma is considered at present work. The plasma electron density irregularities are described by spatial spectrum of an arbitrary form. The small-angle scattering approximation in the invariant ray coordinates is suggested for analytical investigation of the radiation transfer equation. The approximated solution describing spatial-and-angular distribution of radiation reflected from a plasma layer is obtained. The obtained solution can be applied, for example, to the ionospheric radio wave propagation.

## 1 Introduction

Basic goal of the present work consists in derivation of the transfer equation solution describing spatial-and-angular distribution  $P(\vec{\rho}, \omega)$  of radio radiation reflected from a plane stratified layer of magnetized plasma with random irregularities.

The radiation transfer equation (RTE) in a randomly irregular magnetized plasma was obtained in the work [1] under rather general initial assumptions. In particular, the medium average properties were assumed smoothly varying both in space and in time. In the work [2] the radiation energy balance (REB) equation describing radiation transfer in a plane stratified layer of stationary plasma with random irregularities has been deduced. The invariant ray coordinates, allowing one to take into account by a natural way refraction of waves and to represent the equation in the most simple form, were used there. In the work [3] it was shown that the equation REB is a particular case of the radiation transfer equation obtained in [1] and can be deduced from the latter by means of transition to the invariant ray coordinates. Equation REB, thus, allows one to investigate influence of multiple scattering in a plane stratified plasma layer on the characteristics of radiation. In particular, it enables one to determine the spatial-and-angular distribution of radiation leaving the layer if the source directivity diagram and irregularity spatial spectrum are known. A few effects which require of wave amplitudes coherent summation for their description (for example, phenomenon of enhanced backscattering) are excluded from consideration. However, the multiple scattering effects are much stronger, as a rule. This is particularly true for the ionospheric radio propagation.

The numerical methods of the transfer equation solving developed in the theory of neutron transfer and in the atmospheric optics appear useless for the equation REB analysis. They are adapted, basically, to the solution of one-dimensional problems with isotropic scattering and plane incident wave. In a case of magnetized plasma the presence of regular refraction, aspect-sensitive character of scattering on anisometric irregularities and high dimension of the equation REB (it contains

two angular and two spatial coordinates as independent variables) complicate construction of the effective numerical algorithm for its solving. In this situation it is expedient to solve the equation REB in two stages. The first stage consists of obtaining of the approximated analytical solution allowing one to carry out the qualitative analysis of its properties and to reveal of its peculiarities. At the second stage the numerical estimation methods can be applied to the obtained analytical solution, or methods of the numerical solving of the initial equation taking into account the information obtained at the first stage can be designed. Therefore the problem of obtaining of the equation REB approximated analytical solutions is of interest.

We begin the present paper from a detailed exposition of the invariant ray coordinates concept. Then possibility to use of the small-angle scattering in the invariant coordinates approximation is discussed. Two modifications of the REB equation solution are obtained. The analysis of the obtained solutions concludes the paper.

## 2 Invariant ray coordinates and the radiation energy balance equation

It is convenient to display graphically the electromagnetic wave propagation in a plane-stratified plasma layer with the aid of the Poeverlein construction [4,5]. We shall briefly describe it. Let the Cartesian system of coordinates has axis  $z$  perpendicular and the plane  $xOy$  parallel to the plasma layer. We shall name such coordinate system “vertical”. It is assumed that the vector of the external magnetic field  $\vec{H}$  is situated in the plane  $zOy$ . Module of radius-vector of any point inside of the unit sphere with centrum in the coordinate origin corresponds to the value of refractive index  $n_i(v, \alpha)$ , where  $i = 1$  relates to the extraordinary wave,  $i = 2$  relates to the ordinary one,  $v = \omega_e^2/\omega^2$ ,  $\omega_e^2$  is the plasma frequency,  $\omega^2$  is the frequency of a wave,  $\alpha$  is the angle between radius-vector and magnetic field  $\vec{H}$ . The refractive index surface corresponding to a fixed value of  $v$  and to all possible directions of the radius-vector represents a rotation body about an axis parallel to vector  $\vec{H}$  (see fig. 1).

Convenience of the described construction (in fact, this is an example of coordinate system in space of wave vectors  $\vec{k}$ ) is become evident when drawing the wave trajectory: it is represented by a straight line, parallel to the axis  $z$ . This is a consequence of the generalized Snell law, which also requires of equality of the fall angle and exit angle onto/from a layer ( $\theta$ ), and constantness of the wave vector azimuth angle ( $\varphi$ ). Note, that the crossing point of a wave trajectory with the refractive index surface under given value of  $v$  determines current direction of the wave vector in a layer (it is anti-parallel to a radius-vector) and current direction of the group speed vector (it coincides with the normal to the refractive index surface). The projection of a wave trajectory onto the plane  $xOy$  is a point which radius-vector has module  $\sin \theta$  and its angle with relation to axis  $x$  equals to  $\varphi$ . Thus, the coordinates define completely the whole ray trajectory shape in a plane layer and outside of it and are, in this sense, invariant on this trajectory.

Radiation of an arbitrary point source of electromagnetic waves within the solid angle  $\theta \div \theta + d\theta$ ;  $\varphi \div \varphi + d\varphi$  corresponds to the energy flux in the  $\vec{k}$ -space inside of a cylindrical ray tube parallel to axis  $z$  with cross section  $\sin \theta d(\sin \theta) d\varphi = \sin \theta \cos \theta d\theta d\varphi$ . In case of regular (without random irregularities) plasma layer this energy flux is conserved and completely determined by the source directivity diagram:

$$P(z; \theta, \varphi, \vec{\rho}) = P_0(\theta, \varphi, \vec{\rho}), \quad (1)$$

where  $P$  is energy flux density in the direction determined by angles  $\theta, \varphi$  through the point  $\vec{\rho}$  on some base plane situated outside of the layer parallel to it (in the ionosphere case it is convenient to choose the Earth's surface as the base plane),  $z$  is distance from the base plane (height in the ionosphere case). We shall assume in the present paper that function  $z(v)$  is monotonous in the region of wave propagation and reflection. If random irregularities are absent and source of radiation is point, variable  $\vec{\rho}$  in (1) is superfluous, as the matter of fact, since unequivocal relation between it and angles of arrival of a ray  $\theta, \varphi$  exists. When scattering is present the radiation energy redistributes over angular variables  $\theta, \varphi$  and in space what is described by variable  $\vec{\rho}$ . The value of  $P$  satisfies in this case to the equation of radiation energy balance [2,3]:

$$\begin{aligned} \frac{d}{dz} P(z, \theta, \varphi, \vec{\rho}) = & \int \{ -P(z; \theta, \varphi, \vec{\rho}) \sin \theta \cos \theta C^{-1}(z; \theta, \varphi) \cdot \\ & \cdot \sigma [\alpha_0(\theta, \varphi), \beta_0(\theta, \varphi); \alpha(\theta', \varphi'), \beta(\theta', \varphi')] \sin \alpha(\theta', \varphi') \left| \frac{\partial(\alpha, \beta)}{\partial(\theta', \varphi')} \right| + \\ & + P \left[ z; \theta', \varphi', \vec{\rho} - \vec{\Phi}(z; \theta', \varphi'; \theta, \varphi) \right] \sin \theta' \cos \theta' C^{-1}(z; \theta', \varphi') \cdot \\ & \cdot \sigma [\alpha_0(\theta', \varphi'), \beta_0(\theta', \varphi'); \alpha(\theta, \varphi), \beta(\theta, \varphi)] \sin \alpha(\theta, \varphi) \left| \frac{\partial(\alpha, \beta)}{\partial(\theta, \varphi)} \right| \} d\theta' d\varphi' \end{aligned} \quad (2)$$

$C(z; \theta, \varphi)$  is cosine of a ray trajectory inclination angle corresponding to the invariant angles  $\theta$  and  $\varphi$ ;  $|\partial(\alpha, \beta)/\partial(\theta, \varphi)|$  is Jacobean of transition from angular coordinates  $\theta, \varphi$  to the wave vector polar and azimuth angles  $\alpha$  and  $\beta$  in the "magnetic" coordinate system (which axis  $0z$  is parallel to the magnetic field);

$$[\alpha_0(\theta, \varphi), \beta_0(\theta, \varphi); \alpha(\theta', \varphi'), \beta(\theta', \varphi')] \equiv \sigma[\theta, \varphi; \theta', \varphi']$$

is scattering differential cross section describing intensity of the scattered wave with wave vector coordinates  $\alpha, \beta$  in magnetic coordinate system (corresponding invariant coordinates are  $\theta'$  and  $\varphi'$ ) which arises at interaction of the wave with wave vector coordinates  $\alpha_0, \beta_0$  (invariant coordinates  $\theta$  and  $\varphi$ ) with irregularities. Vector function  $\vec{\Phi}(z; \theta', \varphi'; \theta, \varphi)$  represents the displacement of the point of arrival onto the base plane of a ray which has angular coordinates  $\theta'$  and  $\varphi'$  after scattering at level  $z$  with relation to the point of arrival of an incident ray with angular coordinates  $\theta, \varphi$ . It is essential that in a plane-stratified medium the function  $\vec{\Phi}$  is determined only by smoothed layer structure  $v(z)$  and does not depend on the scattering point horizontal coordinate and also on coordinate  $\vec{\rho}$  of the incident and scattered rays. Note also that ratio  $\vec{\Phi}(z; \theta, \varphi; \theta', \varphi') = -\vec{\Phi}(z; \theta', \varphi'; \theta, \varphi)$  takes place.

It is possible to check up that equation (2) satisfies to the energy conservation law: when integrating over all possible for level  $z$  values of  $\theta, \varphi$  and all  $\vec{\rho}$  its right side turns into zero. It is natural since in absence of true absorption the energy inside the plasma layer does not collected.

Analyzing expression for the scattering differential cross section in a magnetized plasma (see, for example, [6]), it is easy to be convinced that the following symmetry ratio takes place:

$$\sigma[\theta, \varphi; \theta', \varphi'] n^2 \cos \vartheta'_g = \sigma[\theta', \varphi'; \theta, \varphi] n'^2 \cos \vartheta_g \quad (3)$$

where  $\vartheta_g$  is angle between the wave vector and group speed vector,  $n$  is refractive index. Using (3) the equation (2) can be presented as follows:

$$\begin{aligned} \frac{d}{dz} P(z, \vec{\rho}, \theta, \varphi) = & \int Q(z; \theta, \varphi; \theta', \varphi') \\ & \left\{ P(z, \vec{\rho} - \vec{\Phi}(z; \theta', \varphi'; \theta, \varphi), \theta', \varphi') - P(z, \vec{\rho}, \theta, \varphi) \right\} d\theta' d\varphi' \end{aligned} \quad (4)$$

where  $Q(z; \theta, \varphi; \theta', \varphi') = \sigma(\theta, \varphi; \theta', \varphi') C^{-1}(z, \theta, \varphi) \sin \theta' |d\Omega'_k/d\Omega|$ , and quantity  $\tilde{Q}(z; \theta, \varphi; \theta', \varphi') \equiv Q(z; \theta, \varphi; \theta', \varphi') \sin \theta \cos \theta$  is symmetric with relation to rearrangement of shaded and not shaded variables. The equation REB in the form (4) has the most compact and perfect appearance. It is clear from physical reasons that (4) has to have the unique solution for given initial distribution  $P_0(\theta, \varphi, \vec{\rho})$ . The obtained equation can be directly used for numerical calculation of the signal strength spatial distribution in presence of scattering. However, as it was noted at introduction already, this approach leads to essential difficulties. Subsequent sections describe the method of construction of the energy balance equation approximated analytical solution.

### 3 Small-angle scattering approximation in the invariant ray coordinates

Let us consider the auxiliary equation of the following kind, which differs from (4) only by absence of the dash over variable  $\omega$  marked by arrow:

$$\frac{d}{dz} P(z, \vec{\rho}, \omega) = \int Q(z; \omega; \omega') \left\{ P(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega) - P(z, \vec{\rho}, \omega) \right\} d\omega' \quad (5)$$

where designation  $\omega = \{\theta, \varphi\}$ ,  $d\omega = d\theta d\varphi$  has been used for the sake of compactness. Equation (5) can be easily solved analytically by means of Fourier transformation over variable  $\vec{\rho}$ . The solution has the following form:

$$P(z, \vec{q}, \omega) = P_0(\vec{q}, \omega) S(z, 0; \vec{q}, \omega), \quad (6)$$

where  $P_0(\vec{q}, \omega)$  is the Fourier image of the radiation energy flux density passing the layer in absence of scattering and the value of  $S$  is defined by the expression

$$S(z_2, z_1, \vec{q}, \omega) = \exp \left\{ \int_{z_1}^{z_2} dz' \int d\omega' Q(z'; \omega; \omega') \left[ \exp \left( i \vec{q} \vec{\Phi}(z'; \omega; \omega') \right) - 1 \right] \right\} \quad (7)$$

One should note that integration over  $z$  in this and subsequent formulae, in fact, corresponds to integration along the ray trajectory with parameters  $\theta, \varphi$ . The area of integration over  $\omega'$  includes rays which reflection level  $h_r(\omega') > z$ .

Let us transform now equation (4) by the following way:

$$\begin{aligned} \frac{d}{dz} P(z, \vec{\rho}, \omega) = & \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega) - P(z, \vec{\rho}, \omega) \right\} + \\ & + \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega') - P(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega) \right\} \end{aligned} \quad (8)$$

Its solution will be looked for in the form

$$P(z, \vec{\rho}, \omega) = \tilde{P}(z, \vec{\rho}, \omega) + X(z, \vec{\rho}, \omega) \quad (9)$$

Thus, auxiliary equation (5) allows to present the solution of the equation (4) in the form (9). This is an exact representation while some approximated expressions for quantities  $\tilde{P}$  and  $X$  are not used.

By substituting of (9) into the equation (4) one can obtain the following equation for the unknown function  $X$ :

$$\begin{aligned} \frac{d}{dz}X(z, \vec{\rho}, \omega) = & \int d\omega' Q(z; \omega; \omega') \{ \tilde{P}(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega') - \\ & - \tilde{P}(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega) \} + \int d\omega' Q(z; \omega; \omega') \{ X(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega') - \\ & - X(z, \vec{\rho}, \omega) \} \end{aligned} \quad (10)$$

We shall assume now that the most probable distinction of angles  $\omega'$  and  $\omega$  is small. The heuristic basis for this assumption is given by analysis of the Poeverlein construction (fig. 1). It is easy to be convinced examining the Poeverlein construction that scattering near the reflection level even for large angles in the wave vectors space entails small changes of the invariant angles  $\theta, \varphi$ . This is especially true for irregularities strongly stretched along the magnetic field (in this case the edges of scattered waves wave vectors form circles shown in fig. 1 as patterns A and B). One should note also that the changes of invariant angles  $\theta, \varphi$  are certainly small if scattering with small change of a wave vector direction takes place. This situation is typical for irregularity spectra, in which irregularities with scales more than sounding wave length dominate. Thus, the small-angle scattering approximation in the invariant coordinates has wider applicability area than common small-angle scattering approximation.

Scattering with small changes of  $\theta, \varphi$  entails small value of  $|\vec{\Phi}|$ . That follows directly both from sense of this quantity and from the fact what  $|\vec{\Phi}(z, \omega, \omega)| = 0$ . Let us make use of that to carry out expansion of quantity  $X$  at the right side of the equation (10) into the Taylor series with small quantities  $\omega' - \omega$  and  $|\vec{\Phi}|$ . Note that making similar expansion of function  $P$  at the initial equation (4) would be incorrect since function  $P$  may not to have property of continuity. For example, in case of a point source,  $P_0$  is a combination of  $\delta$ -functions. As it will be shown later, the function  $X$  is expressed from  $P_0$  by means of repeated integration and, hence, differentiability condition fulfills much easier for it.

Leaving after expansion only small quantities of the first order, we obtain the following equation in partial derivatives:

$$\frac{\partial}{\partial z}X(z, \vec{\rho}, \omega) - A_\omega(z, \omega) \frac{\partial}{\partial \omega}X(z, \vec{\rho}, \omega) + A_{\vec{\rho}}(z, \omega) \frac{\partial}{\partial \vec{\rho}}X(z, \vec{\rho}, \omega) = f(z, \vec{\rho}, \omega), \quad (11)$$

where

$$A_\omega(z, \omega) = \int d\omega' Q(z; \omega; \omega')(\omega' - \omega);$$

$$A_{\vec{\rho}}(z, \omega) = \int d\omega' Q(z; \omega; \omega') \vec{\Phi}(z, \omega, \omega');$$

$$\begin{aligned} f(z, \vec{\rho}, \omega) = & \int d\omega' Q(z; \omega; \omega') \\ & \left\{ \tilde{P}(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega') - \tilde{P}(z, \vec{\rho} + \vec{\Phi}(z; \omega; \omega'), \omega) \right\} \end{aligned}$$

Here is the characteristic system for the equation (11):

$$\frac{dX}{dz} = f(z, \vec{\rho}, \omega); \quad \frac{d\vec{\rho}}{dz} = A_{\vec{\rho}}(z, \omega); \quad \frac{d\omega}{dz} = -A_\omega(z, \omega),$$

and initial conditions for it at  $z = 0$ :

$$X = 0; \quad \vec{\rho} = \vec{\rho}_0; \quad \omega = \omega_0.$$

It is necessary to emphasize the distinction between quantity  $\vec{\rho}'$ , which is a function of  $z$ , and invariant variable  $\vec{\rho}$ .

Solving the characteristic system we obtain:

$$\omega = \omega(z, \omega_0), \quad \vec{\rho}' = \vec{\rho} - \int_z^{z_0} dz' A_{\vec{\rho}}(z', \omega(z', \omega_0)).$$

where  $z_0$  is  $z$ -coordinate of the base plane. It follows that

$$X(z_0, \vec{\rho}, \omega) = \int_0^{z_0} dz' f \left\{ z', \vec{\rho} - \int_{z'}^{z_0} dz'' A_{\vec{\rho}}[z'', \omega(z'', \omega_0)], \omega(z', \omega_0) \right\} \quad (12)$$

Generally, expression (12) gives the exact solution of the equation (11). However, since we are already within the framework of the invariant coordinate small-angle scattering approximation which assumes small value of  $A_\omega(z, \omega)$ , it is possible to simplify the problem a little. Assuming  $A_\omega \cong 0$  and omitting index 0 at invariant coordinates  $\omega$ , we are coming to the following approximate representation for function  $X$ :

$$X(z_0, \vec{\rho}, \omega) = \int_0^{z_0} dz' \{ \tilde{P} [z, \vec{\rho} + \vec{\Phi}(z'; \omega; \omega') + \vec{D}(z_0, z', \omega), \omega'] - \tilde{P} [z', \vec{\rho} + \vec{\Phi}(z; \omega; \omega') + \vec{D}(z_0, z', \omega), \omega] \} \quad (13)$$

where  $\vec{D}(z_2, z_1, \omega) = \int_{z_1}^{z_2} d\omega' Q(z; \omega; \omega') \vec{\Phi}(z, \omega, \omega')$ .

Thus, in the invariant coordinate small-angle scattering approximation the solution of the equation REB (4) is represented as a sum of two terms (see (9)), the first of which is

$$\tilde{P}(z, \vec{\rho}, \omega) = \frac{1}{(2\pi)^2} \int d^2 q P_0(\vec{q}, \omega) \cdot \exp \left\{ i\vec{q}\vec{\rho} + \int_0^{z_0} dz' \int d\omega' Q(z'; \omega; \omega') \left[ \exp \left( i\vec{q}\vec{\Phi}(z'; \omega; \omega'), \omega \right) - 1 \right] \right\} \quad (14)$$

where  $\frac{1}{(2\pi)^2} \int d^2 q P_0(\vec{q}, \omega) \exp(i\vec{q}\vec{\rho}) = P_0(\vec{\rho}, \omega)$ , and the second one is given by expression (13).

The solution can be presented in the most simple form if one uses again the smallness of quantity  $|\vec{\Phi}|$  and expands the second exponent in the formula (14) into a series. Leaving after expansion only small quantities of the first order, one can obtain:

$$P(z, \vec{\rho}, \omega) \cong P_0 \left[ \vec{\rho} + \vec{D}(z_0, 0, \omega), \omega \right] + \int_0^{z_0} dz' \int d\omega' Q(z'; \omega; \omega') \cdot \{ \tilde{P} [z, \vec{\rho} + \vec{\Phi}(z'; \omega; \omega') + \vec{D}(z_0, z', \omega) + \vec{D}(z', 0, \omega'), \omega'] - \tilde{P} [z', \vec{\rho} + \vec{\Phi}(z; \omega; \omega') + \vec{D}(z_0, 0, \omega), \omega] \} \quad (15)$$

The last operation is the more precise the faster value of  $P_0(\vec{q}, \omega)$  decreases under  $|\vec{q}| \rightarrow \infty$ . The solution of the radiation energy balance equation obtained in the present section in the form (9), (14), (13), or in the form (15), expresses the spatial-and-angular distribution of radiation intensity passing layer of plasma with scattering through the spatial-and-angular distribution of the incident radiation, that is, in essence, through the source directivity diagram.

## 4 Alternative approach in solving the REB equation

The REB equation solving method stated in the previous section is based on representation of quantity  $P(z, \vec{\rho}, \omega)$  as a sum of the singular part  $\tilde{P}(z, \vec{\rho}, \omega)$  and the regular one  $X(z_0, \vec{\rho}, \omega)$ . Regularity of the  $X(z_0, \vec{\rho}, \omega)$  has allowed one to use the expansion into the Taylor series over variables  $\vec{\rho}$  and  $\omega$  at the equation (10) right side and to transform the integral-differential equation (10) into the first order partial derivative differential equation (11).

However, the stated approach is not the only possible. The REB equation can be transformed right away using Fourier-representation of the function  $P(z, \vec{\rho}, \omega)$  :

$$P(z, \vec{\rho}, \omega) = \frac{1}{(2\pi)^2} \int d^2q P(z, \vec{q}, \omega) \exp(i\vec{q}\vec{\rho}) \quad (16)$$

Substitution of (16) into (4) gives the following equation for quantity  $P(z, \vec{\rho}, \omega)$  :

$$\frac{d}{dz} P(z, \vec{q}, \omega) = \int d\omega' Q(z; \omega; \omega') \left\{ P(z, \vec{q}, \omega') \exp\left(i\vec{q}\vec{\Phi}(z; \omega; \omega')\right) - P(z, \vec{q}, \omega) \right\} \quad (17)$$

The quantity  $P(z, \vec{q}, \omega)$  is a differentiable function even when  $P(z, \vec{\rho}, \omega)$  has some peculiarities. Therefore, in the invariant coordinate small-angle scattering approximation it is possible to use the following expansion:

$$P(z, \vec{q}, \omega') \cong P(z, \vec{q}, \omega) + \frac{\partial P(z, \vec{q}, \omega)}{\partial \omega} (\omega' - \omega). \quad (18)$$

Substituting (18) in (17) we obtain the partial derivative differential equation

$$\frac{\partial}{\partial z} P(z, \vec{q}, \omega) - \tilde{A}(z, \vec{q}, \omega) \frac{\partial}{\partial \omega} P(z, \vec{q}, \omega) - P(z, \vec{q}, \omega) \tilde{S}(z, \vec{q}, \omega) = 0, \quad (19)$$

where

$$\begin{aligned} \tilde{S}(z, \vec{q}, \omega) &= \int d\omega' Q(z'; \omega; \omega') \left[ \exp\left(i\vec{q}\vec{\Phi}(z'; \omega; \omega'), \omega\right) - 1 \right] \\ \tilde{A}(z, \vec{q}, \omega) &= \int d\omega' Q(z'; \omega; \omega') \exp\left(i\vec{q}\vec{\Phi}(z'; \omega; \omega'), \omega\right) (\omega' - \omega). \end{aligned}$$

The characteristic system

$$\frac{d\omega}{dz} = -\tilde{A}(z, \vec{q}, \omega), \quad \frac{dP}{dz} = \tilde{S}(z, \vec{q}, \omega) P(z, \vec{q}, \omega) \quad (20)$$

with initial conditions  $P = P_0(\vec{q}, \omega), \omega = \omega_0$  at  $z = 0$  has the following solution:

$$P(z, \vec{\rho}, \omega) = \frac{1}{(2\pi)^2} \int d^2q P_0(\vec{q}, \omega_0) \exp\left\{i\vec{q}\vec{\rho} + \int_0^z dz' \tilde{S}[z', \vec{q}, \omega(z', \vec{q}, \omega_0)]\right\} \quad (21)$$

This solution of the REB equation turns into the expression (14) for  $\tilde{P}$  when  $\tilde{A}(z, \vec{q}, \omega) \rightarrow 0$ . But the latter limit transition corresponds to the invariant coordinate small-angle scattering approximation used in the previous section under derivation of (13) and subsequent expressions. Let us note, however, that in (21), in contrast with (9), any additional terms do not appear. It allows one to assume that in used approximation the ratio

$$X(z, \vec{\rho}, \omega) \ll P(z, \vec{\rho}, \omega) \quad (22)$$

is fulfilled. Additional arguments to the benefit of this assumption will be presented in the following section.

## 5 Analysis of the solution of the REB equation

We shall show, first of all, that the obtained solution satisfies to the energy conservation law. For this purpose it is necessary to carry out integration of the left and right sides of (15) over  $\omega$  and  $\vec{\rho}$  multiplied them previously by  $\sin \theta \cos \theta$ . The area of integration over angles is defined by the condition that both wave  $\omega$  and wave  $\omega'$  achieve the same level  $z$  (since at level  $z$  their mutual scattering occurs). To satisfy this condition one should add factors  $\Theta[h_r(\omega) - z]$  and  $\Theta[h_r(\omega') - z]$  to the integrand expression, where  $\Theta(x)$  is Heviside step function,  $h_r(\omega)$  is the maximum height which can be reached by a ray with parameters  $\theta, \varphi$ . Now integration can be expanded over all possible values of angles, i.e., over interval  $0 \div \pi/2$  for  $\theta$  and over interval  $0 \div 2\pi$  for  $\varphi$ . Then, (15) becomes

$$\int P(\omega) \sin \theta \cos \theta d\omega = \int P_0(\omega) \sin \theta \cos \theta d\omega + \int_0^z dz' \int d\omega' \Theta[h_r(\omega) - z'] \Theta[h_r(\omega') - z'] \tilde{Q}(z'; \omega, \omega') [P_0(\omega') - P_0(\omega)]$$

where  $P(\omega), P_0(\omega)$  is a result of integration of  $P(z_0, \vec{\rho}, \omega)$  and  $P_0(\vec{\rho}, \omega)$  correspondingly over variable  $\vec{\rho}$ .

Due to antisymmetry of the integrand expression with relation to rearrangement of shaded and not shaded variables, the last term in (??) is equal to zero. Thus, equation (??) reduces to

$$\int P(z_0, \vec{\rho}, \omega) \sin \theta \cos \theta d\omega d^2 \rho = \int P_0(\vec{\rho}, \omega) \sin \theta \cos \theta d\omega d^2 \rho \quad (23)$$

expressing the energy conservation law: the radiation energy full flux through the plane remains constant regardless of scattering, as it should be in case of real (dissipative) absorption absence. It is not difficult to check that parity (23) is valid for the exact solution in the form (9) and also for the solution in the form (21).

With relation to the solution in the form (9) the carried out discussion discovers one curious peculiarity. It appears that the radiation energy complete flux through the base plane is determined by the first term  $\left(\tilde{P}\right)$ . The second one ( $X$ ) gives zero contribution to the energy complete flux.

Let us investigate in more detail the structure of quantity  $X(z, \vec{\rho}, \omega)$  in the invariant coordinate small-angle scattering approximation. Proceeding to the Fourier-representation in the expression (13) produces

$$X(z_0, \vec{q}, \omega) = \int_0^{z_0} dz' \int d\omega' Q(z'; \omega, \omega') \left[ \tilde{P}(z', \vec{q}, \omega') - \tilde{P}(z', \vec{q}, \omega) \right] \exp \left\{ i\vec{q} \left[ \vec{\Phi}(z; \omega; \omega') + \vec{D}(z_0, 0, \omega) \right] \right\}$$

Employing regularity of function  $\tilde{P}(z, \vec{q}, \omega)$ , the last expression can be written as

$$X(z_0, \vec{q}, \omega) = \int_0^{z_0} dz' \frac{\partial \tilde{P}(z, \vec{q}, \omega)}{\partial \omega} \tilde{A}(z', \vec{q}, \omega) \exp \left[ i\vec{q} \vec{D}(z_0, z', \omega) \right]$$

where quantity  $\tilde{A}(z, \vec{q}, \omega)$  is defined by (19). Thus, it becomes evident that limit transition  $\tilde{A}(z, \vec{q}, \omega) \rightarrow 0$  entails also  $X(z_0, \vec{\rho}, \omega) \rightarrow 0$ . This property has



been established in section 4 with the aid of comparison of two variants of the REB equation solution. Now we see that its presence is determined by structure of quantity  $X(z_0, \vec{\rho}, \omega)$ .

Results of the present section give the weighty ground to believe that the radiation spatial-and-angular distribution is determined basically by the first term in the solution (9). The second term represents the amendment to the solution which can be neglected in the invariant coordinate small-angle scattering approximation. This statement validity can be checked under detailed research of properties of the obtained REB equation approximated solutions by numerical methods.

## 6 Conclusion

In the present work the heuristic basis for use of the invariant coordinate small-angle scattering approximation is considered under solving of the RTE for a magnetized plasma layer. Within the framework of this approximation two versions of the analytical solution have been obtained. They describe spatial-and-angular distribution of radiation reflected from a monotonous plasma layer with small-scale irregularities.

The final physical conclusions about influence of the multiple scattering effects in a layer of plasma on the spatial-and-angular characteristics of radiation are possible on the basis of detailed numerical research of the obtained solutions. Such research is a subject of other our works.

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